Effect of Reynolds Number in Laminar Flow Through a Sudden Planar Contraction

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The effect of inertia forces in flows through sudden contractions have been studied extensively for axisymmetric geometries. Both experimental and numerical results have been obtained for a wide range of Reynolds numbers and a variety of contraction ratios (Rama et al., 1971; Christiansen et al., 1972, Kestin et al., 1973; Vrentas and Duda, 1973; Kim-E et al., 1983). Many aspects of axisymmetric flow in contractions have been discussed recently in a review by Boger (1982) that encompasses Newtonian and non-Newtonian viscoelastic fluids. It has been found that the role of inertia is to decrease the size of the reservoir corner vortex and increase the entrance correction, i.e., the excess pressure drop over and above the pressure drop that would result if the fluid were in fully developed flow in each of the circular tubes.

In viscometry the flow of viscous materials takes place both through capillary and slit dies. It is therefore of interest to also examine two-dimensional flows in planar domains. It is the purpose of this note to provide the corresponding results from numerical calculations on Newtonian fluids for various *Re* numbers in planar sudden contractions.

MATHEMATICAL FORMULATION AND METHOD OF SOLUTION

The Navier-Stokes equations for two-dimensional flow of an incompressible fluid are

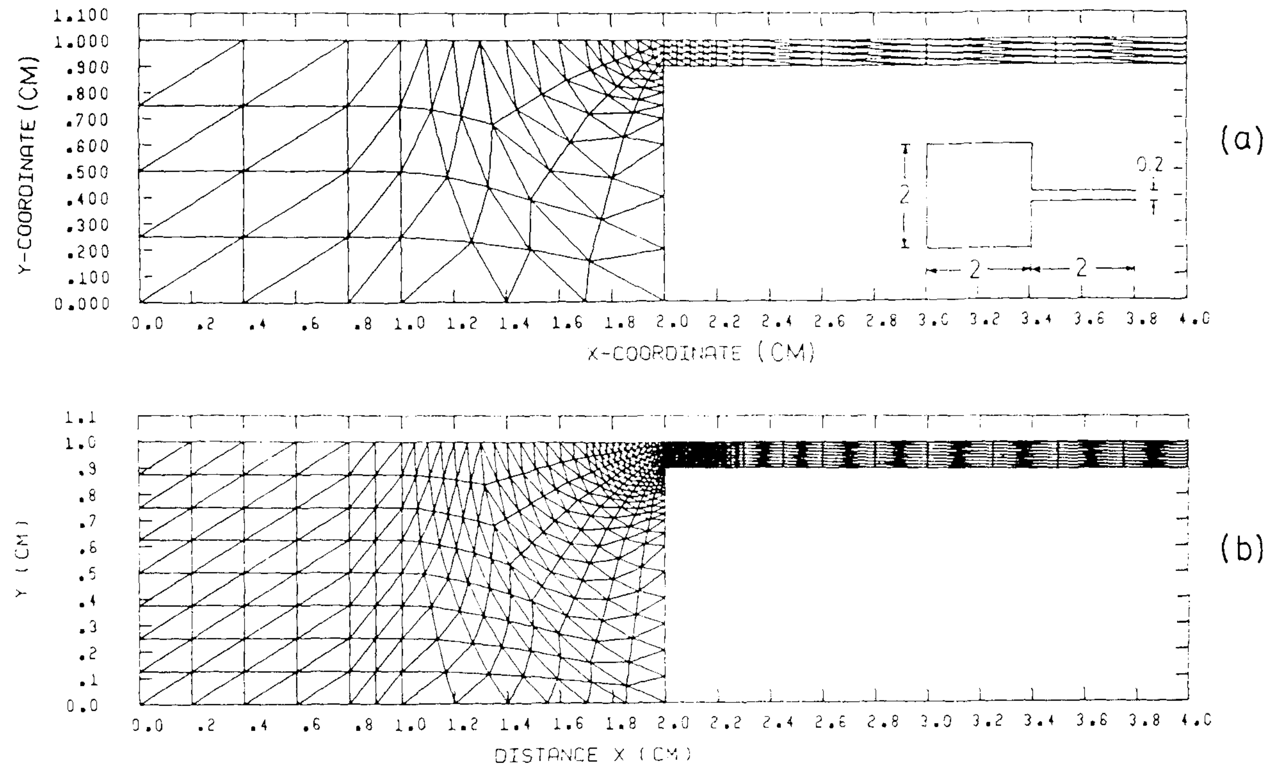

Figure 1. Finite element grid and die dimensions (inset). (*a*) Grid for velocities-pressure (200 elements, 459 nodes). (*b*) Grid for stream function (800 elements, 459 nodes).

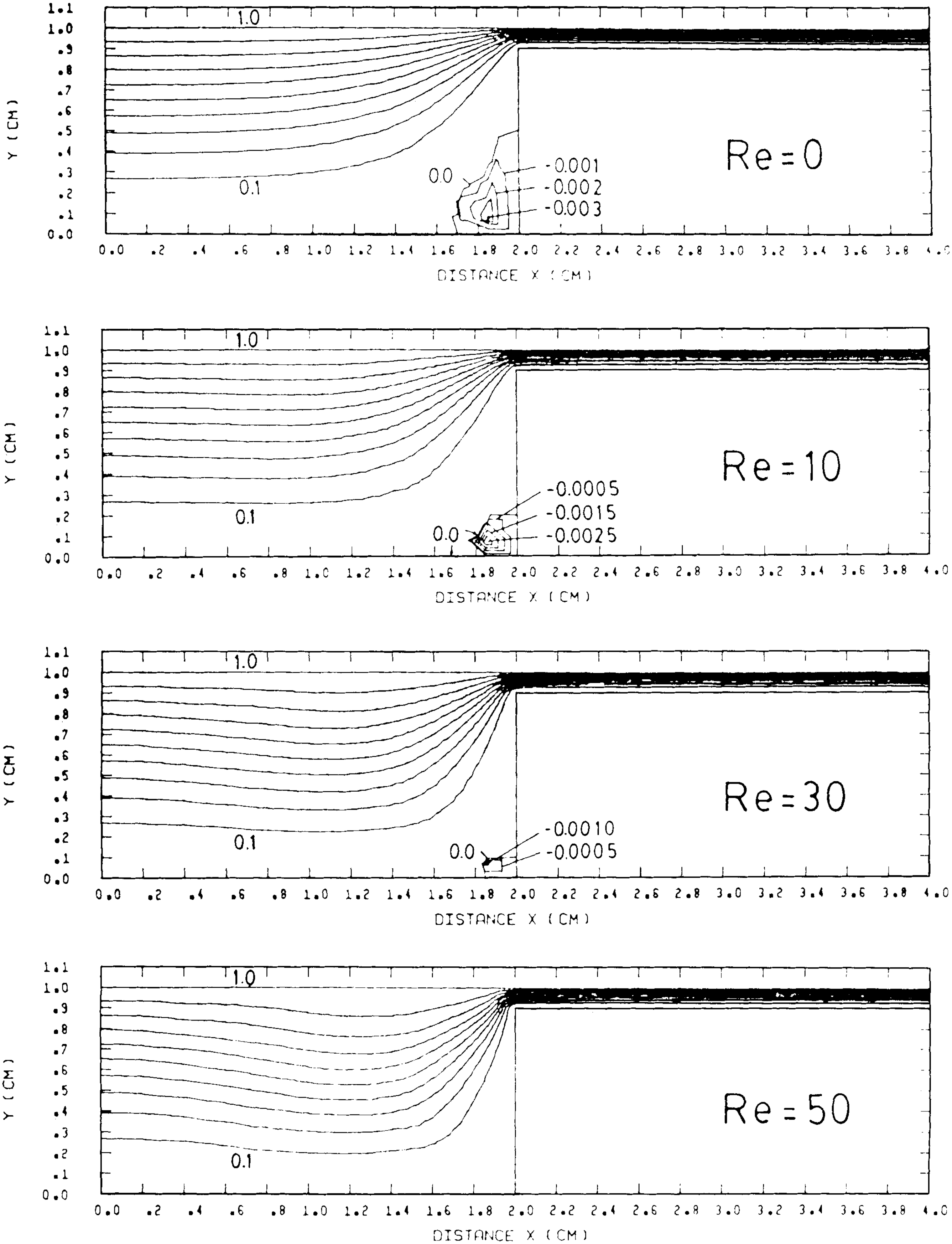

Figure 2. Flow patterns of a Newtonian fluid for different *Re* numbers (10:1 planar sudden contraction).

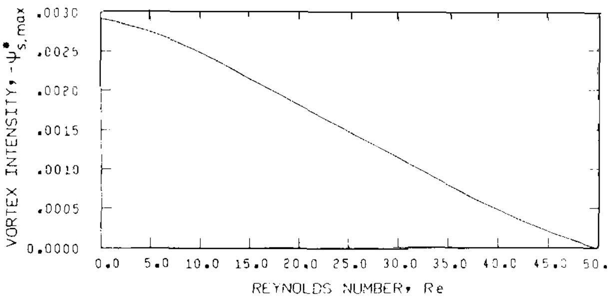

Figure 3. Vortex intensity vs. *Re* number (Newtonian fluid).

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \tag{1}$$

$$\rho\left(v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y}\right) = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} \tag{2}$$

$$\rho\left(v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y}\right) = -\frac{\partial p}{\partial y} + \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} \tag{3}$$

where ρ is the density.

For a Newtonian fluid, we have

$$\tau_{xx} = \mu \frac{\partial v_x}{\partial x} \tag{4a}$$

$$\tau_{yy} = \mu \frac{\partial v_y}{\partial y} \tag{4b}$$

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right) \tag{4c}$$

where μ is a constant viscosity.

The above differential equations are cast in an integral form using the Galerkin method (Huebner and Thornton, 1982). The finite element method is then employed for the solution of a discretized set of equations using as primitive variables the velocities v_x, v_y, and pressure p. The domain is subdivided into triangular elements with quadratic variation assumed for the velocities and linear for the pressure. The inertia terms of Eqs. 2 and 3 give rise to a nonsymmetric stiffness matrix (see, e.g., Gartling et al., 1977), and the resulting nonlinear set of algebraic equations is solved iteratively using direct substitution (Picard method). Convergence becomes more difficult to achieve as the Reynolds number increases, and finally divergence occurs at a certain value depending on the numerical scheme used, the contraction ratio, and the finite element grid.

RESULTS AND DISCUSSION

The finite element calculations were carried out using our MACVIP finite element program (Mitsoulis et al., 1983) as modified to account for noncreeping flows. The results presented here are for a 10:1 planar contraction. For such a contraction ratio the reservoir pressure drop is negligible. It can thus be assumed that excess pressure drop results are also reasonably good approximations for entry flow from an infinite reservoir. The flow domain is shown in Figure 1 (inset) along with the finite element grids for velocities-pressure and stream function formulations. Due to symmetry only half the domain need be considered. Many elements are concentrated near the entrance to the slit to better capture the changes in the velocity gradients (Mitsoulis et al., 1984). The entrance and exit lengths were taken long enough to justify the imposition of fully developed velocity profiles at both ends. The boundary conditions also include no slip at the wall ($v_x = v_y = 0$) and symmetry at the centerline ($v_y = 0$). The pressure is specified to be zero at one point (here the exit wall).

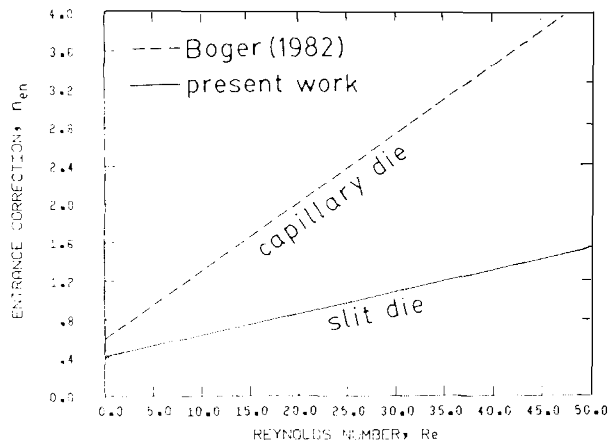

Figure 4. Entrance correction vs. *Re* number (Newtonian fluid).

Runs were carried out for a series of Reynolds numbers (Re). The Reynolds number is defined as

$$Re = \frac{\rho \overline{V} 2H}{\mu} \tag{5}$$

where $\overline{V}$ is the average velocity for fully developed flow in the slit and $2H$ is the slit width. The solution process starts from the inertialess Stokes field ($Re = 0$). Successive solutions are obtained by incrementing ρ (and therefore Re). Some representative flow patterns for various Re numbers are shown in Figure 2. The streamlines have been obtained a posteriori from the velocity field by solving the Poisson equation for the stream function Ψ using the finite element grid of Figure 1b. The stream function has been normalized to take values between 0 (wall) and 1 (centerline). It is evident that inertia forces serve to decrease the corner vortex both in size and intensity. This is in agreement with the results presented for axisymmetric geometries. The corner vortex disappears for $Re \geq 50$. It is interesting to note the bending of the streamlines toward the reservoir walls as Re increases.

The above findings are also illustrated in Figure 3, where the vortex intensity (maximum value of the dimensionless stream function $-\Psi^*_{s,max}$ of the secondary flow) is plotted against Re. The value of $-\Psi^*_{s,max}$ corresponds to the percentage of flow rate that recirculates in the reservoir corner.

The total pressure drop in the system (ΔP) is used to evaluate the entrance correction n_{en} (or equivalent entrance length L_{eq}; see Kim-E et al., 1983)

$$n_{en} = \frac{\Delta P - (\Delta P_{res} + \Delta P_{s\ell})}{2\tau_w} \tag{6}$$

where ΔP_{res} and ΔP_{sl} are the pressure drops for fully developed Poiseuille flow in the reservoir and the slit, respectively, and τ_w is the shear stress at the slit wall. The results are presented in Figure 4 and compared with results from axisymmetric calculations (Boger, 1982, as quoted by Kim-E et al., 1983). It was found that the entrance correction for Newtonian fluids in planar contractions follows the equation

$$n_{en} = 0.0225Re + 0.407 \tag{7}$$

The corresponding equation for axisymmetric geometries is given by Boger (1982) as

$$n_{en} = 0.0709Re + 0.589 \tag{8}$$

A general trend is thus established of increasing pressure loss with Re. This increase is always higher for capillary than slit dies.

NOTATION

- H = half slit width, m
- L_{eq} = equivalent entrance length, dimensionless
- n_{en} = entrance correction, dimensionless
- p = pressure, Pa
- Re = Reynolds number, dimensionless
- $\bar{V}$ = average velocity, m/s
- v_x = velocity in x direction, m/s
- v_y = velocity in y direction, m/s

Greek Letters

- ΔP = total pressure drop, Pa
- ΔP_{res} = pressure drop in the reservoir, Pa
- ΔP_{sl} = pressure drop in the slit, Pa
- μ = viscosity, Pa·s
- ρ = density, kg/m^3
- τ = stress, Pa
- τ_w = wall shear stress, Pa
- Ψ = stream function, m^2/s
- Ψ^* = normalized stream function, dimensionless
- $\Psi^*_{s,\max}$ = maximum intensity of secondary flow, dimensionless

Subscripts

- en = entrance
- eq = equivalent
- res = reservoir
- sl = slit
- $s,\max$ = maximum value of secondary flow
- w = wall
- x = x coordinate
- y = y coordinate

Superscripts

- $\bar{\ }$ = average value
- $*$ = normalized value

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